$\Delta(A) = \sup\{|x-y| : x, y \in A\}$ diameter.

$$\mu_{\alpha,\varepsilon}(A) = \inf_{(U_k)_{k\in\mathbb{N}} \in R_\varepsilon} \sum \Delta(U_k)^\alpha$$

where R_ε is the set of all countable covers of A by sets with diameter $< \varepsilon$.

If $\varepsilon' > \varepsilon$, then any ε cover is a ε' cover, so there are more ε' covers and

$$\mu_{\alpha,\varepsilon'}(A) \leq \mu_{\alpha,\varepsilon}(A). \quad (\text{increasing as } \varepsilon \downarrow 0)$$

$\Rightarrow$ $\lim_{\varepsilon \downarrow 0} \mu_{\alpha,\varepsilon}(A) = \mu_\alpha(A)$ is well defined.

1) If $A \subset \bigcup_{n=1}^{\infty} A_n$, let $(U_{k,n})_{k=1}^{\infty}$ be an ε-cover such that

$$\sum_{k=1}^{\infty} \Delta(U_{k,n})^\alpha < \mu_{\alpha,\varepsilon}(A_n) + \frac{\delta}{2^n} \quad (\delta \text{ is fixed beforehand})$$

Then $\bigcup_{k,n} U_{k,n}$ is an ε cover of A, and

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \Delta(U_{k,n})^\alpha < \sum_{n=1}^{\infty} \mu_{\alpha,\varepsilon}(A_n) + \delta$$

This only makes sense if $\sum_{n=1}^{\infty} \mu_\alpha(A_n) < \infty$, but there is no harm is assuming that.

This implies $\sum_{n=1}^{\infty} \mu_{\alpha,\varepsilon}(A_n) < \infty$ as well

$\Rightarrow \qquad \mu_{\alpha,\varepsilon}(A) \leq \sum_{n=1}^{\infty} \mu_{\alpha,\varepsilon}(A_n)$ (since δ was arbitrary)

let $\lim_{\varepsilon \uparrow 0}$ and use the monotone convergence theorem on the RHS to get

$$\mu_\alpha(A) \leq \sum_{n=1}^{\infty} \lim_{\varepsilon \downarrow 0} \mu_{\alpha,\varepsilon}(A_n) = \sum_{n=1}^{\infty} \mu_\alpha(A_n)$$

2.) Show $\exists$ $\dim(A) \in [0,d]$ st $\forall\ \alpha > 0$ $\mu_\alpha(A) = \begin{cases} 0 & \alpha > \dim(A) \\ \infty & \alpha < \dim(A) \end{cases}$

Assume first that A is bounded, and $\alpha > d$.

If A is contained inside the $[-L,L]^d$ box, cover it with $(-\varepsilon,\varepsilon)^d$ boxes, one needs at most $(2L/\varepsilon)^d$ many of them, and their diameter is $2\sqrt{d}\,\varepsilon$.

Thus $\mu_{\alpha,\varepsilon}(A) \leq \left(2L/\varepsilon\right)^d 2\left(\sqrt{d}\right)^\alpha \varepsilon^\alpha = C\,\varepsilon^{\alpha-d}$. Let $\varepsilon \downarrow 0$

to show $\mu_\alpha(A) = 0$

- Suppose $\alpha < \alpha'$ Then $\forall$ ε cover $(U_n)_{n=1}^{\infty}$ of A, we have

$$\sum \Delta(U_n)^\alpha \geq \sum \Delta(U_n)^{\alpha'} \qquad \text{if } \Delta(U_n) \leq \varepsilon < 1$$

So its clear that μ_α is decreasing (nonincreasing).

Let $\dim(A) = \inf\{\alpha \leq d : \mu_\alpha(A) = 0\}$.

Suppose $\alpha > \dim(A)$, by definition $\mu_\alpha(A) = 0$.
Suppose $0 < \alpha < \dim(A)$. Suppose $\mu_\alpha(A) < \infty$.

Let $\alpha < \alpha' < \dim(A)$. For any ε, $\mu_{\alpha,\varepsilon}(A) \leq \mu_\alpha(A) < \infty$.
Choose a cover st

$$\sum_{n=1}^{\infty} \Delta(U_n)^\alpha < \mu_{\alpha,\varepsilon}(A) + \delta$$

Then $\mu_{\alpha',\varepsilon}(A) \leq \sum_{n=1}^{\infty} \Delta(U_n)^{\alpha'} \leq \varepsilon^{\alpha'-\alpha} \sum_{n=1}^{\infty} \Delta(U_n)^\alpha < \varepsilon^{\alpha'-\alpha}(\mu_{\alpha,\varepsilon}(A) + \delta)$

$$\leq \varepsilon^{\alpha'-\alpha}(\mu_\alpha(A) + \delta).$$

Let $\varepsilon \downarrow 0$, and this shows $\mu_{\alpha'}(A) = 0$. But this contradicts the defn of $\dim(A)$.

3) $\mu(B) \leq r^\alpha$. Let U be s.t. $|\Delta(U)| = r$. Then,

$U \subseteq B(x,r)$ where $x \in U$. If $\exists y \in U$ st $|x-y| > r$
then this is a contradiction.

$\Rightarrow \mu(U) \leq \mu(B(x,r)) \leq r^\alpha$.

Take an ε open cover of $A \subset \bigcup_{n=1}^{\infty} U_n$

$$0 < \mu(A) \leq \sum \mu(U_n) \leq \sum_{n=1}^{\infty} |\Delta(U)|^\alpha$$

monotonicity

Taking infimum we get

$$0 < \mu(A) \leqslant \mu_{\alpha,\varepsilon}(A)$$ Then take $\varepsilon \downarrow 0$

$\Rightarrow \quad \mu_{\alpha,\varepsilon}(A) > 0 \quad \Rightarrow \quad \dim(A) \geqslant \alpha$.